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Structural Synthesis with Reliability Constraints Using Approximation Concepts

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A method for structural synthesis with reliability constraints under service conditions is presented. Uncertain structural parameters and variables are considered. Constraints are imposed on system reliability. Optimization is carried out by generating and solving a sequence of explicit approximate problems. An example problem illustrates the methodology set forth.

Introduction

OPTIMIZATION techniques have been widely accepted as a viable methodology for engineering design. However, failure modes, structural parameters, and design variables are traditionally considered deterministic, which may increase the failure probability over that of unoptimized structures. Uncertainties can be allowed for using reliability theory[1–3] to achieve a balance between cost and safety. Combining reliability procedures and optimization techniques creates a powerful tool to obtain practical designs. The most efficient technique for deterministic optimization is that of approximation concepts,[4] which greatly reduces the number of analyses for convergence. In this method, the solution of the original problem is replaced by the solution of a sequence of explicit approximate problems. The aim of this work is to extend the idea of approximation concepts to the case in which structural parameters and/or design variables are modeled as random variables.

Problem Statement

For a structural design problem, serviceability constraints are given in terms of bounds for displacements, stresses, frequencies, etc. Thus, the set of safe or feasible designs is

$$\Omega = \{(y, p) \mid g_j(y, p) \leq 0, \; j = 1, \ldots, m\} \tag{1}$$

where $y_i, i = 1, \ldots, n$, are the design variables and $p_j, j = 1, \ldots, N$, are structural parameters, both of them modeled as random variables. The functions $g_j(y, p)$ are structural responses that represent the failure modes considered. The synthesis problem is expressed as an optimization problem of the form

$$\begin{aligned} &\text{Min} \quad f(y, p) \\ &\text{s.t.} \quad \text{Prob}[(y, p) \in \Omega] \geq \bar{P} \end{aligned} \tag{2}$$

where $\bar{P}$ is a specified reliability for the system. The main computational effort in solving this problem is in the evaluation of the functions $g_j(y, p)$ for an instance (y, p), which requires a structural analysis and the estimation of the probability, which in turn requires the numerical evaluation of a multiple integral. In this work, parameters and design variables are assumed to be independent Gaussian variables with mean values and standard derivations given by $(\bar{p}_j, \sigma_{p_j})$ and $(\bar{y}_i, \sigma_{y_i})$, respectively. The actual design variables are $\bar{y}_i$ with standard deviations $\sigma_{y_i} = \gamma_i \bar{y}_i$ (γ_i is the coefficient of variation).

Reliability Evaluation

The basic approaches to estimating reliability are the safety-index approach[5–7] and Monte Carlo simulation, which is more accurate, but is computationally very expensive.[8]

Safety-Index Approach

Considering a single constraint in Eq. (1), $g(y, p) \leq 0$, and linearizing it about $(\bar{y}, \bar{p})$ gives

$$\begin{aligned} g(y, p) \approx g(\bar{y}, \bar{p}) &+ \sum_{i=1}^{n} \frac{\partial g}{\partial y_i}(\bar{y}, \bar{p})(y_i - \bar{y}_i) \\ &+ \sum_{j=1}^{N} \frac{\partial g}{\partial p_j}(\bar{y}, \bar{p})(p_i - \bar{p}_i) \leq 0 \end{aligned} \tag{3}$$

This linearized constraint is Gaussian, with mean value and standard deviation given by

$$\mu_g = g(\bar{y}, \bar{p}) \tag{4a}$$

$$\sigma_g^2 = \sum_{i=1}^{n} \left(\frac{\partial g}{\partial y_i}(\bar{y}, \bar{p}) \sigma_{y_i} \right)^2 + \sum_{j=1}^{N} \left(\frac{\partial g}{\partial p_j}(\bar{y}, \bar{p}) \sigma_{p_j} \right)^2 \tag{4b}$$

Defining the safety index $\beta = \mu_g / \sigma_g$, which represents the distance from the origin in the normalized space to the linearized constraint, the probability that the constraint is satisfied is given by

$$P_g = \text{Prob}[g(y, p) \leq 0] \approx \Phi(-\beta) \tag{5}$$

where $\Phi(\cdot)$ is the univariate (0,1) normal cumulative distribution. This estimate is only approximate for nonlinear constraints, and its accuracy is proportional to the curvature of g about $(\bar{y}, \bar{p})$. Here $(\bar{y}, \bar{p})$ can be chosen as the mean value or, for the nonlinear functions, the most probable point for improved accuracy.[6,7] For multiple constraints, the system reliability is estimated by neglecting the correlation between failure modes as

$$\begin{aligned} P_T &= \text{Prob}[\{(x, p) \mid g_j(y, p) \leq 0, j = 1, \ldots, m\}] \\ &\approx \prod_{j=1}^{m} P_j = \prod_{j=1}^{m} \Phi(-\beta_j) \end{aligned} \tag{6}$$

where β_j is the safety index for constraint j.

Monte Carlo Simulation

If $\phi(y, p)$ denotes the joint probability density for (y, p), the system reliability is given by

$$P_T = \int_{\Omega} \phi(y, p) \, \mathrm{d}y \, \mathrm{d}p \tag{7}$$

To estimate this integral numerically, Monte Carlo simulation is the most accurate method, but computationally very intensive.

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The method generates M random points (y, p) according to their distributions, and estimates the integral as[8]

$$\hat{P}_T = \frac{\text{no. of points in } \Omega}{M} \tag{8}$$

The confidence of the estimate can be approximated by one standard derivation as

$$\Delta(\hat{P}_T) = \sqrt{\hat{P}_T(1 - \hat{P}_T)/M} \tag{9}$$

and the simulation is terminated when $\Delta(\hat{P}_T) \leq \varepsilon_p$, where ε_p represents a desired accuracy.

Approximation Concepts

In an optimization context, the system reliability has to be evaluated several times before an optimal design can be obtained. Thus, Monte Carlo simulation is impractical for moderate-sized problems. The safety-index approach is less intensive computationally, but the estimate is poor for correlated nonlinear failure modes.

An efficient alternative is to use the approximation-concept approach in which the functions $g_j(y, p)$ are approximated about a base design to construct explicit high-quality approximations $\tilde{g}_j(y, p)$. With these approximate functions, Ω is approximated ($\tilde{\Omega}$) explicitly in both variables and parameters. Nonlinear approximations for $g_j(y, p)$ are based on intermediate variables and intermediate response quantities to enhance accuracy over a larger region.[9,10] The approximate reliability constraint is then given by

$$\text{Prob}[(y, p) \in \tilde{\Omega}] \geq \bar{P} \tag{10}$$

Failure modes are approximated separately; thus, all correlations are captured. Safety index or Monte Carlo can then be used to evaluate the reliability of the approximated system. Since now $\tilde{\Omega}$ is explicit, both methods are computationally feasible.

The next section illustrates the proposed methodology for static problems, though the procedure can be applied to general systems.

Static Problems

The equilibrium equations for a linear system subject to static loads is given by

$$[K]\{u\} = \{p\} \tag{11}$$

The vector $\{p\}$ is assumed to have the form

$$\{p\} = \sum_{i=1}^{N_p} \{P\}^{(i)} \eta_i \tag{12}$$

the vectors $\{P\}^{(i)}$ are deterministic, whereas the parameters η_i are modeled as normal variables with parameters $(\mu_{\eta_i}, \sigma_{\eta_i})$. The displacements are then given by

$$\{u\} = \sum_{i=1}^{N_p} \{u\}^{(i)} \eta_i \tag{13}$$

$$[K]\{u\}^{(i)} = \{P\}^{(i)}, \qquad i = 1, \ldots, N_p \tag{14}$$

Since the stresses are linear functions of the nodal displacements, they are given by

$$\{\sigma\} = \sum_{i=1}^{N_p} \{\sigma\}^{(i)} \eta_i \tag{15}$$

where $\{\sigma\}^{(i)}$ are the element stresses associated to displacement $\{u\}^{(i)}$. Serviceability constraints are expressed as $\underline{u}_j \leq u_j \leq \bar{u}_j$, $j \in J$, $\underline{\sigma}_i \leq \sigma_i \leq \bar{\sigma}_i$, $i \in I$, where J is the set of constrained degrees of freedom and I is the set of constrained stress points. The allowable stresses are also random with parameters $(\mu_{\underline{\sigma}_i}, \sigma_{\underline{\sigma}_i})$ and $(\mu_{\bar{\sigma}_i}, \sigma_{\bar{\sigma}_i})$. The synthesis problem has the form

$$\text{Min} \quad W(\bar{y}) \tag{16a}$$

$$\text{s.t.} \quad P_T \geq \bar{P}_T \tag{16b}$$

where $W(\bar{y})$ is the expected weight. To approximate Ω, it is only necessary to approximate $u_j^{(i)}$ and $\sigma_k^{(i)}$, since its dependence on η, $\bar{\sigma}$, and $\underline{\sigma}$ is explicit. Reciprocal approximations are used for $u_j^{(i)}$, and for $\sigma_k^{(i)}$ the element forces are approximated linearly (see Ref. 9).

Example Problem

A 10-bar truss problem (Fig. 1) has been chosen to illustrate the method set forth. The mean values and standard deviations for the parameters are $\bar{P}_1 = 100$ ksi, $\bar{P}_2 = 50$ ksi, $\bar{\sigma}_{\text{allow}} = \pm 25$ ksi, $\sigma_{p_1} = 7.5$ ksi, $\sigma_{p_2} = 2.5$ ksi, and $\sigma_{\sigma_{\text{allow}}} = 1250$ psi. Displacement constraints ($|u_y| \leq 5$ in.) are imposed for nodes 1 and 2. The design variables are the truss areas with a standard deviation of 5%. Figure 2 shows the system reliability at the base design ($A_i = 5$ in.^{2}) as a function of the variance of the random variables. For the range considered, Monte Carlo using approximations is more accurate than using the safety index, since all correlations are considered.

Problem 1—Deterministic Design Variables

Cross-sectional areas are considered deterministic. The problem was solved for system reliabilities of $P_T = 0.99$ and $P_T = 0.90$ using safety index and Monte Carlo on the approximate model. The final designs are given in Table 1 (the deterministic case uses mean properties).

The designs using the two approaches are very similar, but in all cases, Monte Carlo gives a lower objective. This can be explained by observing the reliability for the designs in Table 1. These reliabilities were estimated using Monte Carlo simulation on the exact model. It is observed that the exact reliability constraint is not critical for the safety index approach, which is a consequence of the error in the reliability estimation. Since now $\tilde{\Omega}$ is explicit, the better accuracy

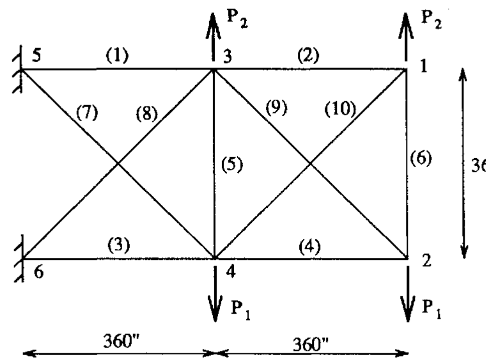

Fig. 1 Ten-bar truss: $E = 10^7$ psi and $\rho = 0.1$ pci.

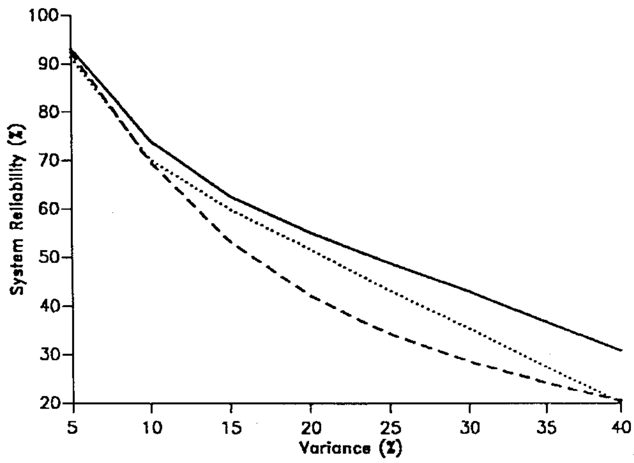

Fig. 2 System reliability: random cross-sectional areas: ——, exact; ·····, approximations; and – – –, safety index.

Table 1 **Optimal design, problem 1**

	Area, in.^{2}				
		Monte Carlo		Safety index	
Element	Deterministic	$P_T = 0.99$	$P_T = 0.90$	$P_T = 0.99$	$P_T = 0.90$
1	2.05	3.44	3.03	4.19	4.08
2	0.10	1.38	0.66	1.11	0.99
3	5.95	7.06	6.65	6.63	5.89
4	1.97	2.42	2.61	2.04	2.51
5	0.10	1.70	1.46	2.02	1.66
6	2.03	2.77	2.60	3.77	2.44
7	5.59	6.10	5.83	5.70	4.99
8	0.10	1.37	1.15	2.03	2.40
9	2.78	3.24	3.04	2.84	3.06
10	0.10	0.89	0.84	1.34	0.99
Weight, lb	875.43	1267.22	1165.36	1318.06	1214.36
Reliability, %	2.3	99.0	90.0	99.6	94.2
No. of analyses	6	9	6	7	6

Table 2 **Optimal design, problem 2**

	Area, in.^{2}		
Element	Deterministic	$P_T = 0.99$	$P_T = 0.95$
1	3.02	5.01	3.57
2	0.10	1.37	1.54
3	6.79	8.79	8.29
4	2.80	2.88	2.67
5	0.10	1.07	8.83
6	2.00	3.07	3.11
7	5.43	5.84	6.97
8	0.43	1.86	1.62
9	3.96	4.33	4.21
10	0.10	1.25	1.52
Weight, lb	1041.33	1474.55	1451.11
Reliability, %	1.9	99.0	95.0
No. of analyses	13	8	9

using Monte Carlo does not justify the use of the safety index on account of efficiency considerations.

Problem 2—Random Design Variables

In this second problem, the cross-sectional areas are also considered random. Two runs were made using Monte Carlo simulation with required system reliabilities of $P_T = 0.99$ and $P_T = 0.95$, respectively. The final designs are given in Table 2, from which it is seen that the number of analyses is similar to Problem 1.

Conclusions

An efficient methodology for optimal design with random variables and/or parameters has been presented. The problem is solved using approximations based on the use of intermediate response quantities and intermediate variables to approximate the safe space. Numerical results show the efficiency of the proposed technique, with near-optimal solutions obtained within six to nine analyses. Thus, the use of approximation concepts not only proves to be efficient, but allows one to consider all failure modes simultaneously. Numerical results show that the safety index approach and Monte Carlo simulation show similar behavior. The safety index is attractive because of its analytical properties and could be used sequentially, as a second level of approximation, within each approximate problem. Monte Carlo simulation can always be used to obtain an accurate estimate of the system reliability at the end of each design cycle.

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